# MATH 320 NOTES 2

So, far we know that if $A \in M_{n,n}(F)$, the following are equivalent:

(1) $A$ is invertible.
(2) $rank(A) = n$.
(3) $Ax = \mathbf{0}$ has only the trivial solution.
(4) $Ax = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

Next we will define **the determinant** of a matrix, $\det(A) \in F$, and show that the above hold iff $\det(A) \neq 0$. So, computing the determinant will be one more way of deciding if $A$ is invertible.

## Section 4.1 Determinants of order 2

**Definition 1.** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *The determinant of* $A$ *is* $\det(A) = ad - bc$.

Some remarks:

(1) $\det(I_2) = 1$, the determinant of the zero matrix is 0.
(2) If $A$ has a zero row or a zero column, $\det(A) = 0$.
(3) If the row of $A$ are multiples of each other, then $\det(A) = 0$. That's because if $a = kc, b = kd$, we have $ad - bc = kcd - kdc = 0$.

It turns out that the converse of the last item is also true:

**Theorem 2.** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then* $A$ *is invertible iff* $\det(A) \neq 0$.

*Proof.* The easy direction: If $A$ is not invertible, then $rank(A) < 2$, so the rows are linearly dependent, so they are multiples of each other, and so by the above note, $\det(A) = 0$.

Now for the harder direction: If $A$ is invertible, then $rank(A) = 2$, and so $A$ cannot have a zero row. So, $a \neq 0$ or $b \neq 0$ (or both). Suppose $a \neq 0$ (the other case is similar). Then by the type 3 elementary row operation, $R_2 - \frac{c}{a}R_1$, we obtain the matrix

$$B = \begin{pmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{pmatrix}$$

Since elementary row operations preserve the rank, we have that $rank(B) = rank(A) = 2$, and so $B$ is invertible. Then $B$ cannot have a row of zeros. It follows that $d - \frac{cb}{a} \neq 0$, and so $da \neq cb$. Then $\det(A) = ad - cb \neq 0$. $\square$

**Lemma 3.** *Suppose* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is invertible. Then* $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

*Proof.* Calculate $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = I_2$ □

As a function $\det : M_{2,2}(F) \to F$ is *not* a linear transformation, but it is something close.

**Lemma 4.** *The function* $\det : M_{2,2}(F) \to F$ *is a linear function of each row, when the other one is fixed:*

- $\det \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{pmatrix} = \det \begin{pmatrix} a_1 & b_1 \\ c & d \end{pmatrix} + \det \begin{pmatrix} a_2 & b_2 \\ c & d \end{pmatrix}$.
- $\det \begin{pmatrix} ka & kb \\ c & d \end{pmatrix} = k \ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

*Similarly, if we fix the first row.*

### Section 4.2 Determinants of order $n$.

First let us introduce some notation. Let $A \in M_{n\times n}(F)$. We will denote the $(i,j)$-th entry by $a_{ij}$. Also, given $1 \leq i, j \leq n$, let $\overline{A}_{ij} \in M_{n-1\times n-1}(F)$ be the submatrix obtained by removing the $i$-th row and the $j$-th column of $A$.

**Definition 5.** *Let* $A \in M_{n\times n}(F)$. *If* $n = 1$, $\det(A) = A = a_{11}$. *If* $n > 1$, *then*

$$\det(A) = \Sigma_{k=1}^{n}(-1)^{k+1} a_{1k} \cdot \det(\overline{A}_{1k}).$$

Note that this definition is by induction on $n$. I.e. we assume we know the definition of determinant of dimension $(n-1)\times(n-1)$, and use it to define the determinant in the case of dimension $n \times n$. Also, the above formula is computing the determinant of $A$ along *the first row.* Later we will see that we can compute it along any row or column.

**Exercise:** Verify that for $n = 2$, the above formula gives the same definition as in the last section.

The next lemma is the generalization of Lemma 4 for $n$ by $n$ matrices.

**Lemma 6.** *The function* $\det : M_{n\times n}(F) \to F$ *is a linear function of each row, when the other ones are fixed. More precisely,*

$$\det \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} + d\mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix} = \det \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix} + d \det \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}.$$

*Proof.* The proof is by induction on $n$. If $n = 1$, it is clear, so suppose $n > 1$.

Let $A = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} + d\mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}$; $B = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}$; $C = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}$

We want to show that $\det(A) = \det(B) + d\det(C)$. There are two cases.

**Case 1.** $r = 1$. Then, by our definition, $\det(A) =$

$$\Sigma_{k=1}^{n}(-1)^{k+1}a_{1k} \cdot \det(\overline{A}_{1k}) = \Sigma_{k=1}^{n}(-1)^{k+1}(b_{1k} + dc_{1k}) \cdot \det(\overline{A}_{1k}) =$$

$$\Sigma_{k=1}^{n}(-1)^{k+1}b_{1k} \cdot \det(\overline{A}_{1k}) + d\Sigma_{k=1}^{n}(-1)^{k+1}c_{1k} \cdot \det(\overline{A}_{1k}) = \det(B) + d\det(C)$$

Note that here for every $k$, $\overline{A}_{1k} = \overline{C}_{1k} = \overline{B}_{1k}$, because the only difference in the matrices $A, B$, and $C$ is the the first row.

**Case 2**. $r > 1$. Then $\det(A) = \Sigma_{k=1}^{n}(-1)^{k+1}a_{1k} \cdot \det(\overline{A}_{1k})$ and by the inductive hypothesis, for each $k$,

$$\det(\overline{A}_{1k}) = \det(\overline{B}_{1k}) + d\det(\overline{C}_{1k}).$$

This is because, the submatrices have dimension $(n-1) \times (n-1)$, and the $(r-1)$-th row of $\overline{A}_{1k}$ equals the $(r-1)$-th row of $\overline{B}_{1k}$ plus $d$ times the $(r-1)$-th row of $\overline{C}_{1k}$. Plugging in, we have,

$\det(A) =$

$= \Sigma_{k=1}^{n}(-1)^{k+1}a_{1k} \cdot (\det(\overline{B}_{1k}) + d\det(\overline{C}_{1k})) =$

$= \Sigma_{k=1}^{n}(-1)^{k+1}a_{1k} \cdot \det(\overline{B}_{1k}) + d\Sigma_{k=1}^{n}(-1)^{k+1}a_{1k} \cdot \det(\overline{C}_{1k}) =$

$= \det(B) + d\det(C)$. □

**Corollary 7.** *If $A$ has a row of zeros, then* $\det(A) = 0$.

*Proof.* Say the $r$-th row of $A$ has only zeros, i.e. this row is $\mathbf{0} = 0 \cdot \mathbf{0}$. Then by the above lemma applied to row $r$, we have that $\det(A) = 0 \cdot \det(A) = 0$. □

Our next goal is to show that we can compute the determinant by expanding along any row. First we show it in the simplest case – when the row in question is of the form $e_k$, for some $k$, i.e. the vector with 1 in the $k$-th coordinate and 0s everywhere else.

**Lemma 8.** *Let $A \in M_{n\times n}(F)$ and suppose that the $r$-th row of $A$ is $e_k$. Here $1 \leq k, r \leq n$ and $1 < n$. Then* $\det(A) = (-1)^{r+k}\det(\overline{A}_{rk})$.

*Proof.* By induction on $n$. For $n = 2$, it is an exercise to verify it.

Suppose $n > 2$, and we have the result for smaller dimensions. Again we divide it into two cases.

**Case 1**. $k = 1$. Then $a_{1k} = 1$ and for all $j \neq k$, $a_{1j} = 0$. So by the formula for the determinant, we have $\det(A) = (-1)^{1+k}\det(\overline{A}_{1k}) = (-1)^{r+k}\det(\overline{A}_{rk})$.

**Case 2.** $k > 1$. Then by the inductive hypothesis for each submatrix $\overline{A}_{1j}$,

- if $j = k$, the $r-1$-th row is $\mathbf{0}$ (because we have removed the $k$-th column on $A$). Then $\det(\overline{A}_{1k}) = 0$ by the above corollary.
- if $j \neq k$, the $r-1$-th row is $e_k$ (with one less dimension). Then, by induction,
  (1) $\det(\overline{A}_{1j}) = (-1)^{r-1+k-1}\det(C_{rj})$, if $j < k$
  (2) $\det(\overline{A}_{1j}) = (-1)^{r-1+k}\det(C_{rj})$, if $j > k$
  where $C_{rj}$ is the submatrix of $\overline{A}_{1j}$ by removing the $r$-th row and $j$-th column of $\overline{A}_{1j}$.

When we plug this information if the formula for the determinant, after some computation, we get the desired result. For details, see pg 214 in the textbook. □

Now we can finally prove a very useful fact about determinants: the we can compute them by expanding along *any* row of $A$.

**Theorem 9.** *Let $A \in M_{n\times n}(F)$, and let $1 \leq i \leq n$. We can compute* $\det(A)$ *by expanding along row $i$ as follows:* $\det(A) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ik}\cdot\det(\overline{A}_{ij})$.

*Proof.* Denote the $i$-th row of $A$ by $a_i = \langle a_{i1}, a_{i2}, ..., a_{in}\rangle = a_{i1}e_1 + ... + a_{in}e_n$. Fir each $j \leq n$, let $B_j \in M_{n\times n}(F)$ be the matrix obtained by replacing $a_i$ with $e_j$, i.e. $A$ and $B_j$ differ only in row $i$. (For example the $i$th row of $B_1$ is $\langle 1, 0, ..., 0\rangle$ and every other row is like in $A$.)

By Lemma 8, we have that for each $j$, $\det(B_j) = (-1)^{i+j}\det(\overline{(B_j)_{ij}})$. Since $\overline{(B_j)_{ij}}$ is obtained from $B_j$ by removing the $i$-th row and the $j$-column. Since the only difference between $A$ and each $B_j$ is in the $i$-th row, it follows that $\overline{(B_j)_{ij}} = \overline{A}_{ij}$. Plugging in, we get

$$\det(B_j) = (-1)^{i+j}\det(\overline{A}_{ij}).$$

Then, by linearity (Lemma 6), we have that

$$\det(A) = \Sigma_{j=1}^{n}a_{ij}\det(B_j) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\overline{A}_{ij}).$$

□

Using the above theorem, next we will show what effect doing elementary row operations have on the determinant of a matrix.

**Lemma 10.** *(Elementary row operations and the determinant) Suppose $A \in M_{n\times n}(F)$ and $B$ is obtained from $A$ by doing one elementary row operation.*

(1) *(Type 1) If $B$ is obtained from $A$ by interchanging two rows, then* $\det(B) = -\det(A)$.

(2) *(Type 2) If $B$ is obtained from $A$ by multiplying one row by $k$, then* $\det(B) = k\det(A)$.
(3) *(Type 3) If $B$ is obtained from $A$ by adding a multiple of one row to another, then* $\det(B) = \det(A)$.

*Proof.* **Part 1**. By induction on $n$. If $n = 2$, it is a straightforward calculation. Suppose that we interchange rows $r$ and $k$. Let $i < n$, $i \neq r, i \neq k$. Expanding along row $i$, we get, $\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij})$.

Now, for each $j$, $\overline{B}_{ij}$, is obtained from $\overline{A}_{ij}$ by interchanging rows $r$ and $k$, and so by induction, $\det(\overline{B}_{ij}) = -\det(\overline{A}_{ij})$. Also, since the $i$-th row of $A$ and $B$ are the same, we have that $a_{ij} = b_{ij}$ for all $j \leq n$. So,

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij}) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}(-\det(\overline{A}_{ij})) =$$
$$-\Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\overline{A}_{ij}) = -\det(A).$$

**Part 2.** Suppose that $B$ is obtained by multiplying row $i$ by $k$. Expanding along row $i$, we have that

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij}) =$$

$$\Sigma_{j=1}^{n}(-1)^{i+j}ka_{ij}\det(\overline{A}_{ij}) = k\Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\overline{A}_{ij}) = k\det(A).$$

Here $\overline{A}_{ij} = \overline{B}_{ij}$, because we only changed row $i$.

**Part 3.** Again, this is by induction on $n$. Suppose $B$ is obtained from $A$ by adding a multiple of row $r$ to row $k$. If $n = 2$, this can be verified directly. Otherwise, let $i \leq n, i \neq r, i \neq k$. Then for all $j \leq n$, $\overline{B}_{ij}$ is obtained from $\overline{A}_{ij}$ by adding the same multiple of of row $r$ to row $k$. So, by induction, $\det(\overline{B}_{ij}) = \det(\overline{A}_{ij})$. Expanding along row $i$, we have that

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij}) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\overline{A}_{ij}) = \det(A).$$

□

The following is an immediate corollary:

**Corollary 11.** *If $A, B \in M_{n\times n}(F)$ are such that $B$ is obtained from $A$ by doing elementary row operations, then* $\det(B) = 0$ *iff* $\det(A) = 0$.

**Theorem 12.** *Let $A \in M_{n\times n}$. Then* $\det(A) = 0$ *iff $rank(A) < n$ iff $A$ is not invertible.*

*Proof.* We already know that $rank(A) < n$ iff $A$ is not invertible from previous sections. So, we just have to show this is equivalent to $\det(A) = 0$.

For one direction, suppose $rank(A) < n$. Then its rows must be linearly dependent, so there is some row, say row $i$, which can be written as a linear combination of the other rows. Then by doing type 3 elementary row operations, adding multiples of other rows to row $i$, we obtain a matrix $B$ from $A$, such that the $i$-th row of $B$ is all zeros. Then $\det(B) = 0$. But since doing type 3 elementary row operations don't change the determinant, we also have $\det(A) = 0$.

For the other direction, suppose that $rank(A) = n$. Then by doing elementary row operations, we can obtain $I_n$ from $A$. Since $\det(I_n) = 1 \neq 0$, then $\det(A) \neq 0$.

□

**Section 4.3 A couple of more properties of the determinant.**

We start with two lemmas that follow from the effect of doing elementary row operations on the determinant.

**Lemma 13.** *Suppose that $E$ is an elementary row matrix.*

(1) *(Type 1) If $E$ is obtained by interchanging two rows of $I_n$, then* $\det(E) = -1$*;*
(2) *(Type 2) If $E$ is obtained from $I_n$ by multiplying a row by $k$, then* $\det(E) = k$*;*
(3) *(Type 3) If $E$ is obtained from $I_n$ by adding a multiple of one row to another, then* $\det(E) = 1$*;*

*Proof.* The proof is immediate using Lemma 9 and that $\det(I_n) = 1$. □

**Lemma 14.** *Suppose that $A = EB$, where $E$ is an elementary (row) matrix, then* $\det(A) = \det(E) \cdot \det(B)$

*Proof.*

(1) (Type 1) If $E$ is obtained by interchanging two rows of $I_n$, then $\det(E) = -1$ and $A$ is obtained from $B$ by interchanging the same two rows. So
$$\det(A) = -\det(B) = \det(E) \cdot \det(B);$$
(2) (Type 2) If $E$ is obtained from $I_n$ by multiplying a row by $k$, $\det(E) = k$ and $A$ is obtained from $B$ by multiplying the same row by $k$. So
$$\det(A) = k \det(B) = \det(E) \cdot \det(B);$$
(3) (Type 3) If $E$ is obtained from $I_n$ by adding a multiple of one row to another, then $\det(E) = 1$ and $A$ is obtained from $B$ by same operation. So
$$\det(A) = \det(B) = \det(E) \cdot \det(B);$$

□

And now, for the main theorem about matrix multiplication and the determinant:

**Theorem 15.** *Suppose that $A, B \in M_{n \times n}(F)$. Then*
$$\det(AB) = \det(A) \cdot \det(B).$$

Note that this also means that $\det(AB) = \det(BA)$, although of course in general $AB \neq BA$.

*Proof.* **Case 1** $\det(A) = 0$. Then $rank(AB) \leq rank(A) < n$, and so $\det(AB) = 0$. The. $\det(AB) = 0 = \det(A) \cdot \det(B)$.

**Case 2** $\det(A) \neq 0$. Then $A$ is invertible. And so it is the products of elementary matrices. (We can assume these are row elementary). Write $A = E_1 ... E_k$ where each $E_i$ is elementary. Then

$$\det(AB) = \det(E_1 \cdot ...E_k \cdot B) = \det(E_1)\det((E_2 \cdot ...E_k \cdot B) = ...$$

$$\det(E_1)\det(E_2) \cdot ... \det(E_k) \cdot \det(B) = \det(E_1 \cdot ...E_k)\det(B) = \det(A)\det(B).$$

□

**Corollary 16.** *If $A$ is invertible,* $\det(A^{-1}) = \frac{1}{\det(A)}$.

*Proof.* Exercise. □

Finally, we note that the lemmas about elementary row operations and the determinant also hold for column operations. I.e. we have:

**Fact 17.** *Suppose that $B$ is obtained from $A$ by doing one elementary column operation. Say $B = AE$, where $E$ is an elementary column matrix. Then, if $E$ is of:*

(1) *Type 1, interchanging two rows:* $\det(E) = -1$, $\det(B) = -1\det(A)$*;*
(2) *Type 2, multiplying a row by $k$:* $\det(E) = k$, $\det(B) = k\det(A)$*;*
(3) *Type 3:* $\det(E) = 1$, $\det(B) = \det(A)$.

*In particular, if $E$ is an elementary matrix (row or column), then*

$$\det(E) = \det(E^t).$$

We leave the proof as an exercise.

**Lemma 18.** $\det(A^t) = \det(A)$.

*Proof.* If $A$ is not invertible, then $\det(A) = 0$, and $n > rank(A) = rank(A^t)$, so $A^t$ is not invertible and $\det(A^t) = 0$.

If $A$ is invertible, then $A = E_1 \cdot ... \cdot E_k$, where each $E_i$ is elementary. So

$$A^t = (E_1 \cdot ... \cdot E_k)^t = E_k^t \cdot ... \cdot E_1^t,$$

and for each $i \leq k$, $\det(E_i^t) = \det(E_i)$. Then,

$$\det(A^t) = \det(E_k^t \cdot ... \cdot E_1^t) = \det(E_k^t) \cdot ... \cdot \det(E_1^t) = \det(E_k) \cdot ... \cdot \det(E_1) = \det(A).$$

□